

SUFFICIENT CONDITIONS IN THE TWO-FUNCTIONAL CONJECTURE FOR UNIVALENT FUNCTIONS

DMITRI PROKHOROV

Dedicated to Professor Promarz M. Tamrazov

ABSTRACT. The two-functional conjecture says that if a function f analytic and univalent in the unit disk maximizes $\operatorname{Re} \{L\}$ and $\operatorname{Re} \{M\}$ for two continuous linear functionals L and M , $L \neq cM$ for any $c > 0$, then f is a rotation of the Koebe function. We use the Löwner differential equation to obtain sufficient conditions in the two-functional conjecture and compare the sufficient conditions with necessary conditions.

1. INTRODUCTION

For the class S of functions $f(z) = z + a_2 z^2 + \dots$ analytic and univalent in the unit disk \mathbb{D} , the two-functional conjecture arose from the description of functions $f \in S$ which satisfy two independent so-called \mathcal{D}_n -equations, see, e.g. [8, p.347-351] and references therein. This conjecture says that if a function $f \in S$ maximizes $\operatorname{Re} \{L\}$ and $\operatorname{Re} \{M\}$ for two continuous linear functionals L and M nonconstant on S , $L \neq cM$ for any $c > 0$, then f is a rotation of the Koebe function $k(z) = z(1 - z)^{-2}$.

Each continuous linear functional on the space \mathcal{A} of all analytic functions in \mathbb{D} has the form $L(h) = \sum_{n=0}^{\infty} c_n a_n$, $h(z) = \sum_{n=0}^{\infty} a_n z^n$, for some sequence of complex numbers c_n , $\limsup_{n \rightarrow \infty} |c_n|^{1/n} < 1$, see, e.g. [8, p.280]. The known results [1], [2], [7], [13], see also survey [10], for special cases of the two-functional conjecture are restricted to functionals

$$(1) \quad L(f) = \sum_{n=2}^n \bar{\lambda}_n a_n, \quad \lambda_n \neq 0, \quad \text{and} \quad M(f) = \sum_{n=2}^m \bar{\mu}_n a_n, \quad \mu_m \neq 0.$$

We shall be assuming $m = n$, since the conjecture is true for the case $m \neq n$ if it is proved for $m = n$, [7].

The two-functional conjecture if it is true characterizes an exclusive role of the Koebe function and its rotations both analytically and geometrically. After de Branges [3] proved the Bieberbach conjecture it became clear that the Koebe function $k(z)$ maximizes simultaneously $\operatorname{Re} \{L_j\}$ for $n - 1$ independent continuous linear functionals $L_k = a_k + a_n$, $k = 2, \dots, n - 1$, and $L_n = a_n$. This means geometrically

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that $k(z)$ delivers a boundary point $x^0 = (2, \dots, n)$ to the value set

$$V_n = \{(a_2, \dots, a_n) : f \in S\},$$

and there is at least $(n-1)$ -dimensional set of support hyperplanes for the $(2n-2)$ -dimensional set V_n through x^0 , namely, hyperplanes with normal vectors $\Lambda_n = (0, \dots, 0, 1)$ and $\Lambda_k = (0, \dots, 0, 1, 0, \dots, 0, 1)$, the unit is at the $(k-1)$ -th place. The result due to Bshouty and Hengartner [5] shows that the set of support hyperplanes for V_n through x^0 is exactly $(n-1)$ -dimensional since if at least one of coefficients λ_k in (1) is not real, then $k(z)$ does not maximize $\operatorname{Re} \{L\}$.

So the two-functional conjecture supposes that if a function $f \in S$ maximizes $\operatorname{Re} \{L\}$ and f is different from any rotation of $k(z)$, then there is only one support hyperplane for V_n through a boundary point $x_f \in \partial V_n$ delivered by f .

In the present article we give sufficient conditions for the two-functional conjecture in terms of coefficients $\bar{\lambda}_k, \bar{\mu}_k, k = 2, \dots, n$, and coefficients of an extremal function $f \in S$. We prove the following theorem.

Theorem 1. *Let a function $f(z) = z + a_2 z^2 + \dots$ maximize $\operatorname{Re} \{L\}$ and $\operatorname{Re} \{M\}$ on S where L and M are given by (1), $m = n$. Suppose that the trigonometric polynomial*

$$(2) \quad \operatorname{Re} \left(- \sum_{k=2}^n \sum_{j=1}^{n-k+1} \bar{\lambda}_{j+k-1} j a_j e^{-i(k-1)u} \right)$$

attains its maximum on $[0, 2\pi]$ at $u = \pi$, and

$$(3) \quad \sum_{k=2}^n \sum_{j=1}^{n-k+1} (-1)^k (k-1)^2 \operatorname{Re} (((1-\alpha)\bar{\lambda}_{j+k-1} + \alpha\bar{\mu}_{j+k-1})j a_j) \neq 0, \quad \alpha \in [0, 1].$$

Then

$$(4) \quad \sum_{k=2}^n \sum_{j=1}^{n-k+1} (-1)^k (k-1) \operatorname{Im} (\bar{\lambda}_{j+k-1} j a_j) = \sum_{k=2}^n \sum_{j=1}^{n-k+1} (-1)^k (k-1) \operatorname{Im} (\bar{\mu}_{j+k-1} j a_j).$$

If additionally

$$(5) \quad \sum_{k=2}^n \sum_{j=1}^{n-k+1} (-1)^k (k-1)^{m+1} \operatorname{Im} (\bar{\lambda}_{j+k-1} j a_j) =$$

$$\sum_{k=2}^n \sum_{j=1}^{n-k+1} (-1)^k (k-1)^{m+1} \operatorname{Im} (\bar{\mu}_{j+k-1} j a_j) = 0, \quad m = 1, 2, \dots,$$

then $f(z)$ is the Koebe function $k(z)$.

In Section 2 we introduce elements of the Löwner theory necessary for the proof method. We note there that every extremal function obeys condition (2) of Theorem 1 up to a suitable rotation. This notion is fixed in Remark 1 of Section 3.

Theorem 1 is proved in Section 3 where we remark also that condition (3) of Theorem 1 is not essential.

Section 4 contains necessary conditions in the two-functional conjecture.

2. THE LÖWNER INTERPRETATION OF AN EXTREMAL PROBLEM

A function $f \in S$ is a support point of S if there is a continuous linear functional L , not constant on S , such that f maximizes $\operatorname{Re} \{L\}$ over S . Every support point of S maps \mathbb{D} onto the complement of a single analytic arc extending from a finite point to infinity, see [12, p.149], [4], [8, p.306].

From the other side, a function $f \in S$ which maps \mathbb{D} onto the plane \mathbb{C} minus a single slit Γ can be represented as

$$(6) \quad f(z) = \lim_{t \rightarrow \infty} e^t w(z, t), \quad z \in \mathbb{D},$$

where $w(z, t) = e^{-t}(z + a_2(t)z^2 + \dots)$ is a solution to the equation

$$(7) \quad \frac{dw}{dt} = -w \frac{e^{iu(t)} + w}{e^{iu(t)} - w}, \quad w(z, 0) \equiv z.$$

The driving term $u(t)$ in the Löwner ordinary differential equation (7) is real analytic provided the slit Γ is analytic [6], see also [9]. Let us express an extremal function f for functionals (1) in terms of an optimal driving function u for the system of differential equations generated by the Löwner equation (7). Put

$$a(t) = \begin{pmatrix} a_1(t) \\ a_2(t) \\ \vdots \\ a_n(t) \end{pmatrix}, \quad a^0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad A(t) = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ a_1(t) & 0 & \dots & 0 & 0 \\ a_2(t) & a_1(t) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1}(t) & a_{n-2}(t) & \dots & a_1(t) & 0 \end{pmatrix}.$$

Equate coefficients at the same powers in expansions of both sides in (7) and obtain the system of differential equations

$$(8) \quad \frac{da}{dt} = -2 \sum_{s=1}^{n-1} e^{-s(t+iu)} A^s(t) a(t), \quad a(0) = a^0, \quad a_1(t) = 1,$$

where A^s denotes the s -th power of the matrix $A(t)$. To realize the maximum principle we introduce an adjoint vector $\Psi(t)$,

$$\Psi(t) = \begin{pmatrix} \Psi_1(t) \\ \Psi_2(t) \\ \vdots \\ \Psi_n(t) \end{pmatrix},$$

with complex valued coordinates $\Psi_1, \Psi_2, \dots, \Psi_n$, and the real pseudo Hamilton function

$$(9) \quad H(t, a, \bar{\Psi}, u) = \operatorname{Re} \left\{ -2 \sum_{s=1}^{n-1} e^{-s(t+iu)} (A^s a)^T \bar{\Psi} \right\},$$

where $\bar{\Psi}$ means the vector with complex conjugate coordinates and the upper index T is the transposition sign. To come to the Hamiltonian formulation we require that

the function $\bar{\Psi}$ satisfies the adjoint system of differential equations

$$(10) \quad \frac{d\bar{\Psi}}{dt} = 2 \sum_{s=1}^{n-1} e^{-s(t+iu)} (s+1) (A^T)^s \bar{\Psi}.$$

Suppose that for the functional L given by (1), $\max \operatorname{Re} \{L\}$ on S is delivered by an extremal function $f \in S$. By (6) and (7), f corresponds to a real analytic optimal driving function u . Equations (8) and (10) with the optimal function u produce the optimal trajectory $(a(t), \bar{\Psi}(t))$ corresponding to f and L , $a(\infty) = a$, $\Psi(\infty) = (0, \lambda_2, \dots, \lambda_n)^T$. The necessary optimal condition requires that the optimal function u satisfies the Pontryagin maximum principle, i.e., along the optimal trajectory $(a(t), \bar{\Psi}(t))$ the function $H(t, a, \bar{\Psi}, u)$ as a function of u is maximized by the optimal driving function $u = u(t)$. Hence this optimal u solves the equation

$$(11) \quad H_u(t, a, \bar{\Psi}, u) = \operatorname{Re} \left\{ 2i \sum_{s=1}^{n-1} e^{-s(t+iu)} s (A^s a)^T \bar{\Psi} \right\} = 0, \quad t \geq 0,$$

along the optimal trajectory.

Note that a rotation $f(z) \rightarrow f_\beta(z) := e^{-i\beta} f(e^{i\beta} z)$ of the extremal function f implies the transformation $L \rightarrow L_\beta$ of the functional L defined by coefficients $\lambda_2, \dots, \lambda_n$ according to (1). The functional L_β should be defined by the coefficients $\nu_k := e^{i(k-1)\beta} \lambda_k$, $k = 2, \dots, n$, because f_β maximizes $\operatorname{Re} \{L_\beta\}$ on S , $\operatorname{Re} \{L(f)\} = \operatorname{Re} \{L_\beta(f_\beta)\}$. The function f_β has a representation (6), (7) with the driving function $u(t) - \beta$ provided $u(t)$ corresponds to f . Therefore we can assume without loss of generality that the optimal driving function u satisfies the initial condition $u(0) = \pi$.

The initial value $\bar{\Psi}(0)$ is uniquely determined by $a = a(\infty)$ and $\lambda_2, \dots, \lambda_n$ from (1). The following lemma was proved earlier [11] in another version.

Lemma 1. *Let $a(t)$ and $\Psi(t)$, $\Psi(\infty) = (0, \lambda_2, \dots, \lambda_n)^T$, obey systems (8) and (10). Then*

$$(12) \quad \bar{\Psi}_k(0) = \sum_{j=1}^{n-k+1} \bar{\lambda}_{j+k-1} j a_j, \quad k = 2, \dots, n.$$

Proof. Differentiating (7) with respect to z we have

$$(13) \quad \frac{d}{dt} \left(\frac{1}{e^t w'(z, t)} \right) = \frac{2w(2e^{iu} - w)}{e^t w'(z, t) (e^{iu} - w)^2}, \quad w'(z, 0) = 1,$$

where $w'(z, t)$ is a derivative of $w(z, t)$ with respect to z . Considering the expansion

$$\frac{z}{e^t w'(z, t)} = \sum_{k=1}^{\infty} q_k(t) z^k,$$

we obtain for $q(t) = (q_1(t), \dots, q_n(t))^T$,

$$(14) \quad \frac{dq}{dt} = 2 \sum_{s=1}^{n-1} e^{-s(t+iu)} (s+1) A^s q.$$

We observe that system (14) for q differs from system (10) for $\bar{\Psi}$ only by the transposition sign T at A^s . In order to satisfy the condition $q(\infty) = (0, \bar{\lambda}_2, \dots, \bar{\lambda}_n)^T$ we denote by

$$g(z, t) = \frac{(\bar{\lambda}_n z^2 + \dots + \bar{\lambda}_2 z^n) f'(z)}{e^t w'(z, t)} = \sum_{k=2}^{\infty} c_k(t) z^k$$

and see that $g(z, t)$ obeys the same equation (13) where $1/w'(z, t)$ is substituted by $g(z, t)$. Hence, $c(t) = (c_2(t), \dots, c_{n+1}(t))^T$ obeys system (14) substituting $q(t)$ by $c(t)$. It is evident that $c(\infty) = (\bar{\lambda}_n, \dots, \bar{\lambda}_2, 0)^T$. The difference in the transposition sign in (10) and (14) implies that $\bar{\Psi}_k(t) = c_{n-k+2}(t)$, $k = 2, \dots, n$. It remains to evaluate $\bar{\Psi}_k(0) = c_{n-k+2}(0)$ for

$$(\bar{\lambda}_n z^2 + \dots + \bar{\lambda}_2 z^n) f'(z) = \sum_{k=2}^{\infty} c_k(0) z^k.$$

Straightforward calculations lead to (12) and complete the proof. \square

3. PROOF OF THEOREM 1

Proof of Theorem 1. Suppose that a function $f(z) = z + a_2 z^2 + \dots$ maximizes $\operatorname{Re} \{L\}$ and $\operatorname{Re} \{M\}$ given by (1), $m = n$, on S . In this case f maximizes also $\operatorname{Re} \{(1 - \alpha)L + \alpha M\}$ for all $\alpha \in [0, 1]$. The function f is represented by (6), (7) with a real analytic optimal driving function u in (7). An optimal trajectory $(a(t), \bar{\Psi}(t))$ obeys systems (8) and (10) with the optimal function u . The maximum principle requires that equality (11) for the pseudo Hamilton function $H(t, a, \bar{\Psi}, u)$ holds identically with the optimal $u(t)$ along the optimal trajectory.

Evaluate

$$\frac{\partial H_u}{\partial a_k} \frac{da_k}{dt} := \frac{\partial H_u}{\partial (\operatorname{Re} a_k)} \frac{d(\operatorname{Re} a_k)}{dt} + \frac{\partial H_u}{\partial (\operatorname{Im} a_k)} \frac{d(\operatorname{Im} a_k)}{dt}, \quad k = 2, \dots, n.$$

Put

$$\tilde{H}(t, a, \bar{\Psi}, u) = -2 \sum_{s=1}^{n-1} e^{-s(t+iu)} (A^s a)^T \bar{\Psi}, \quad \operatorname{Re} \tilde{H}(t, a, \bar{\Psi}, u) = H(t, a, \bar{\Psi}, u).$$

Adding the equalities

$$\frac{\partial H_u}{\partial (\operatorname{Re} a_k)} \frac{d(\operatorname{Re} a_k)}{dt} = \operatorname{Re} \left(\frac{\partial \tilde{H}_u}{\partial a_k} \frac{da_k}{dt} \right) \quad \text{and} \quad \frac{\partial H_u}{\partial (\operatorname{Im} a_k)} \frac{d(\operatorname{Im} a_k)}{dt} = -\operatorname{Im} \left(\frac{\partial \tilde{H}_u}{\partial a_k} \frac{da_k}{dt} \right)$$

we obtain using (8) and (10)

$$\frac{\partial H_u}{\partial a_k} = \operatorname{Re} \left(-4i \sum_{s=1}^{n-2} \sum_{j=1}^{n-s-1} e^{-(s+j)(t+iu)} s \frac{\partial}{\partial a_k} ((A^s a)^T) (A^j a)_k \right),$$

where $(A^j a)_k$ is the k -th coordinate of the column vector $A^j a$. This implies that

$$(15) \quad \frac{\partial H_u}{\partial a} \frac{da}{dt} = \operatorname{Re} \left(-4i \sum_{s=1}^{n-2} \sum_{j=1}^{n-s-1} e^{-(s+j)(t+iu)} s(s+1) a^T (A^T)^{s+j} \bar{\Psi} \right).$$

Similarly, for the vector

$$\frac{\partial H_u}{\partial \bar{\Psi}} \frac{d\bar{\Psi}}{dt}$$

with coordinates

$$\frac{\partial H_u}{\partial \bar{\Psi}_k} \frac{d\bar{\Psi}_k}{dt} := \frac{\partial H_u}{\partial(\operatorname{Re} \bar{\Psi}_k)} \frac{d(\operatorname{Re} \bar{\Psi}_k)}{dt} + \frac{\partial H_u}{\partial(\operatorname{Im} \bar{\Psi}_k)} \frac{d(\operatorname{Im} \bar{\Psi}_k)}{dt},$$

we have according to (15)

$$(16) \quad \frac{\partial H_u}{\partial \bar{\Psi}} \frac{d\bar{\Psi}}{dt} = \operatorname{Re} \left(4i \sum_{s=1}^{n-2} \sum_{j=1}^{n-s-1} e^{-(s+j)(t+iu)} s(s+1) a^T (A^T)^{s+j} \bar{\Psi} \right) = -\frac{\partial H_u}{\partial a} \frac{da}{dt}.$$

In the same way, for

$$H_{ut^m}(t, a, \bar{\Psi}, u) = \operatorname{Re} \left(2i(-1)^m \sum_{s=1}^{n-1} e^{-s(t+iu)} s^{m+1} (A^s a)^T \bar{\Psi} \right), \quad m = 1, 2, \dots,$$

we have

$$(17) \quad \frac{\partial H_{ut^m}}{\partial \bar{\Psi}} \frac{d\bar{\Psi}}{dt} = -\frac{\partial H_{ut^m}}{\partial a} \frac{da}{dt}, \quad m = 1, 2, \dots$$

Condition (2) of Theorem 1 means that $H(0, a^0, \bar{\Psi}(0), u)$ attains its maximum on $[0, 2\pi]$ at $u = \pi$. The initial value $\bar{\Psi}(0)$ is given by (12) in Lemma 1. However, the optimal function u preserves its extremal properties when λ_k are substituted by $(1 - \alpha)\lambda_k + \alpha\mu_k$, $k = 2, \dots, n$. In particular, $u = \pi$ is the maximum point of $H(0, a^0, \bar{\Psi}(\alpha, 0), u)$ for the vector $\bar{\Psi}(\alpha, 0)$ with coordinates

$$\bar{\Psi}_k(\alpha, 0) := \sum_{j=1}^{n-k+1} (\bar{\lambda}_{j+k-1} + \alpha(\bar{\mu}_{j+k-1} - \bar{\lambda}_{j+k-1})) j a_j, \quad k = 2, \dots, n, \quad \alpha \in [0, 1].$$

Hence $u(\alpha, 0) = \pi$ is a root of the equation

$$H_u(0, a^0, \bar{\Psi}(\alpha, 0), u) = \operatorname{Re} \sum_{k=2}^n 2i(k-1) e^{-i(k-1)u} \bar{\Psi}_k(\alpha, 0) = 0.$$

So $u(\alpha, 0)$ does not depend on α . Elementary calculations show that the equation

$$\frac{\partial u(\alpha, 0)}{\partial \alpha} = 0, \quad 0 \leq \alpha \leq 1,$$

is equivalent to (4).

Changing the initial value in system (10) from $\bar{\Psi}(0)$ to $\bar{\Psi}(\alpha, 0)$ we preserve the function f and the optimal driving function $u = u(t) = u(\alpha, t)$ in (7) but the adjoint coordinate $\bar{\Psi}(t)$ in the optimal trajectory $(a(t), \bar{\Psi}(t))$ is substituted by $\bar{\Psi}(\alpha, t)$. Differentiate $H_u(t, a, \bar{\Psi}, u)$ in (11) with respect to t along the optimal trajectory $(a(t), \bar{\Psi}(\alpha, t))$. Taking into account (16) we obtain

$$(18) \quad \frac{d}{dt} H_u(t, a(t), \bar{\Psi}(\alpha, t), u(\alpha, t)) = \frac{\partial H_u}{\partial t} + \frac{\partial H_u}{\partial a} \frac{da}{dt} + \frac{\partial H_u}{\partial \bar{\Psi}} \frac{\partial \bar{\Psi}}{\partial t} + \frac{\partial H_u}{\partial u} \frac{\partial u}{\partial t} =$$

$$H_{ut}(t, a(t), \bar{\Psi}(\alpha, t), u(\alpha, t)) + H_{uu}(t, a(t), \bar{\Psi}(\alpha, t), u(\alpha, t)) u_t(\alpha, t) = 0,$$

which gives the formula

$$u_t(\alpha, 0) = - \frac{H_{ut}(0, a^0, \bar{\Psi}(\alpha, 0), \pi)}{H_{uu}(0, a^0, \bar{\Psi}(\alpha, 0), \pi)} =$$

$$- \frac{\sum_{k=2}^n \sum_{j=1}^{n-k+1} (-1)^k (k-1)^2 \text{Im} (\bar{\lambda}_{j+k-1} + \alpha(\bar{\mu}_{j+k-1} - \bar{\lambda}_{j+k-1}) j a_j)}{\sum_{k=2}^n \sum_{j=1}^{n-k+1} (-1)^k (k-1)^2 \text{Re} (\bar{\lambda}_{j+k-1} + \alpha(\bar{\mu}_{j+k-1} - \bar{\lambda}_{j+k-1}) j a_j)},$$

where the denominator does not vanish because of (3). Condition (5) for $m = 1$ implies that for $u(t) = u(\alpha, t)$,

$$(19) \quad u'(0) = u_t(\alpha, 0) = 0.$$

Suppose by induction that for $u = u(t) = u(\alpha, t)$,

$$(20) \quad u^{(p)}(0) = u_{tp}(\alpha, 0) = 0, \quad p = 1, \dots, m-1,$$

and differentiate $H_u(t, a(t), \bar{\Psi}(\alpha, t), u(\alpha, t))$ along the optimal trajectory $m-1$ times. Taking into account (17) - (20) and the inductive formula

$$\frac{d^{m-1}}{dt^{m-1}} H_u(t, a(t), \bar{\Psi}(\alpha, t), u(\alpha, t)) = H_{ut^{m-1}}(t, a(t), \bar{\Psi}(\alpha, t), u(\alpha, t)) +$$

$$\sum_{j=1}^{m-2} R_j(t) u_{tj}(\alpha, t) + H_{uu}(t, a(t), \bar{\Psi}(\alpha, t), u(\alpha, t)) u_{t^{m-1}}(\alpha, t)$$

with inductively evaluated functions $R_j(t)$, $j = 1, \dots, m-2$, we obtain

$$\frac{d^m}{dt^m} H_u(t, a(t), \bar{\Psi}(\alpha, t), u(\alpha, t))_{t=0} = H_{ut^m}(0, a^0, \bar{\Psi}(\alpha, 0), \pi) +$$

$$H_{uu}(0, a^0, \bar{\Psi}(\alpha, 0), \pi) u_{t^m}(\alpha, 0) = 0.$$

This allows us to find $u_{t^m}(\alpha, 0)$,

$$u_{t^m}(\alpha, 0) = - \frac{H_{ut^m}(0, a^0, \bar{\Psi}(\alpha, 0), \pi)}{H_{uu}(0, a^0, \bar{\Psi}(\alpha, 0), \pi)} =$$

$$- \frac{\sum_{k=2}^n \sum_{j=1}^{n-k+1} (-1)^k (k-1)^{m+1} \text{Im} (\bar{\lambda}_{j+k-1} + \alpha(\bar{\mu}_{j+k-1} - \bar{\lambda}_{j+k-1}) j a_j)}{\sum_{k=2}^n \sum_{j=1}^{n-k+1} (-1)^k (k-1)^2 \text{Re} (\bar{\lambda}_{j+k-1} + \alpha(\bar{\mu}_{j+k-1} - \bar{\lambda}_{j+k-1}) j a_j)},$$

where the denominator does not vanish because of (3). Conditions (5) imply that for $u(t) = u(\alpha, t)$,

$$(21) \quad u^{(m)}(0) = u_{t^m}(\alpha, 0) = 0.$$

Thus it follows from (19) - (21) that $u(0) = \pi$ and $u^{(m)}(0) = 0$ for all $m = 1, 2, \dots$. Since $u(t)$ is real analytic, $u(t) = \pi$ identically for $t \geq 0$. This driving function u determines the Koebe function $k(z)$ by (6), (8) which completes the proof of Theorem 1.

Remark 1. Condition (2) of Theorem 1 can be achieved with the help of a suitable rotation of the extremal function f .

Remark 2. Condition (3) of Theorem 1 is not essential. Indeed, the optimal function $u(t)$ maximizes $H(t, a, \overline{\Psi}, u)$ along the optimal trajectory, and $u(0) = \pi$ is the maximum point of the function $H(0, a^0, \overline{\Psi}(0), u)$. Therefore

$$H_u(0, a^0, \overline{\Psi}(0), \pi) = 0 \quad \text{and} \quad H_{uu}(0, a^0, \overline{\Psi}(0), \pi) \leq 0.$$

If $H_{uu}(0, a^0, \overline{\Psi}(0), \pi) = 0$, then there is an even number $m = 2l$, $l > 1$, such that

$$H_{u^q}(0, a^0, \overline{\Psi}(0), \pi) = 0, \quad 2 \leq q \leq 2l - 1, \quad H_{u^{2l}}(0, a^0, \overline{\Psi}(0), \pi) < 0.$$

Since $H(t, a, \overline{\Psi}, u)$ is linear with respect to $\overline{\Psi}$, it is possible to choose a minimal even number which provides the above property for all

$$\overline{\Psi}_k(0) = \sum_{j=1}^{n-k+1} (\overline{\lambda}_{j+k-1} + \alpha(\overline{\mu}_{j+k-1} - \overline{\lambda}_{j+k-1})), \quad k = 2, \dots, n, \quad 0 \leq \alpha \leq 1,$$

with certain $\lambda_2, \dots, \lambda_n, \mu_2, \dots, \mu_n$. In this case we repeat the proof of Theorem 1 changing $H_{uu}(0, a^0, \overline{\Psi}(0), u)$ for $H_{u^{2p}}(0, a^0, \overline{\Psi}(0), u)$.

Corollary 1. Let a function $f(z) \in S$ with real coefficients a_2, \dots, a_n maximize $\operatorname{Re} \{L\}$ and $\operatorname{Re} \{M\}$ for functionals L and M given by (1), $m = n$, with real coefficients $\lambda_2, \dots, \lambda_n$ and μ_2, \dots, μ_n and satisfy the conditions of Theorem 1. Then $f(z)$ is the Koebe function $k(z)$.

Proof. Under the conditions of Corollary 1 the sufficient conditions (5) are trivially verified. \square

4. NECESSARY CONDITIONS IN THE TWO-FUNCTIONAL CONJECTURE

Observe that equality (4) is necessary under the conditions of Theorem 1. Let us adduce another necessary relations which are not too far from the sufficient conditions (5) but lead to the opposite conclusions.

Theorem 2. Let a function $f(z) \in S$ and functionals L and M satisfy the conditions of Theorem 1. Then either conditions (5) are satisfied for all $m \geq 2$ or there are numbers $m \geq 1$ and $c_m \neq 0$ such that conditions (5) are satisfied for all $p < m$ and

$$(22) \quad \operatorname{Im} \sum_{k=2}^n \sum_{j=1}^{n-k+1} (-1)^k (k-1)^{m+1} ((1-\alpha)\overline{\lambda}_{j+k-1} + \alpha\overline{\mu}_{j+k-1}) =$$

$$c_m \operatorname{Re} \sum_{k=2}^n \sum_{j=1}^{n-k+1} (-1)^k (k-1)^2 ((1-\alpha)\overline{\lambda}_{j+k-1} + \alpha\overline{\mu}_{j+k-1}), \quad 0 \leq \alpha \leq 1.$$

In the last case f is not a rotation of the Koebe function $k(z)$.

Proof. Suppose that conditions (5) for f , L and M are satisfied for all $p < m$. It was shown in the proof of Theorem 1 that in this case $u^{(p)}(0) = 0$, $p = 1, \dots, m-1$, and

$$(23) \quad u^{(m)}(0) = u_{tm}(\alpha, 0) = -\frac{H_{ut^m}(0, a^0, \bar{\Psi}(\alpha, 0), \pi)}{H_{uu}(0, a^0, \bar{\Psi}(\alpha, 0), \pi)} =$$

$$(-1)^m \frac{\operatorname{Im} \sum_{k=2}^n \sum_{j=1}^{n-k+1} (-1)^k (k-1)^{m+1} ((1-\alpha)\bar{\lambda}_{j+k-1} + \alpha\bar{\mu}_{j+k-1})}{\operatorname{Re} \sum_{k=2}^n \sum_{j=1}^{n-k+1} (-1)^k (k-1)^2 ((1-\alpha)\bar{\lambda}_{j+k-1} + \alpha\bar{\mu}_{j+k-1})}.$$

As soon as $u_{tm}(\alpha, 0)$ does not depend on $\alpha \in [0, 1]$, we conclude that either the numerator in (23) vanishes or the numerator and the denominator are proportional. The last case is reflected in condition (22) which means that $u_{tm}(\alpha, 0)$ is equal to $(-1)^m c_m \neq 0$. Therefore $u(t)$ does not correspond to any rotation of the Koebe function, and this completes the proof. \square

Note that condition (22) is reduced to (5) if $c_m = 0$. The two-functional conjecture supposes that $c_m = 0$ every time.

Write the proportionality condition in (23) in another way. Equality (22) is equivalent to the system of equations

$$\operatorname{Im} \sum_{k=2}^n \sum_{j=1}^{n-k+1} (-1)^k (k-1)^{m+1} \bar{\lambda}_{j+k-1} j a_j = c_m \operatorname{Re} \sum_{k=2}^n \sum_{j=1}^{n-k+1} (-1)^k (k-1)^2 \bar{\lambda}_{j+k-1} j a_j,$$

$$\operatorname{Im} \sum_{k=2}^n \sum_{j=1}^{n-k+1} (-1)^k (k-1)^{m+1} \bar{\mu}_{j+k-1} j a_j = c_m \operatorname{Re} \sum_{k=2}^n \sum_{j=1}^{n-k+1} (-1)^k (k-1)^2 \bar{\mu}_{j+k-1} j a_j.$$

Example 1. Let

$$L = \lambda a_2 + a_4, \quad \lambda \in \mathbb{R}.$$

For $\lambda \geq 0$, $\operatorname{Re} \{L\}$ is maximized on S by the Koebe function $k(z)$. We will be looking for negative λ so that $\operatorname{Re} \{L\}$ is still locally maximized by $k(z)$. To realize the Löwner approach we apply Lemma 1 and put

$$\Psi_2(0) = 9 + \lambda, \quad \Psi_3(0) = 4, \quad \Psi_4(0) = 1$$

in (8), (10). Then

$$H(0, a^0, \bar{\Psi}(0), u) = -2(\cos 3u + 4 \cos 2u + (9 + \lambda) \cos u) := p_\lambda(u).$$

Denote by $\lambda_0 = -0.931\dots$ the root of the equation

$$(24) \quad 25\lambda^3 + 37\lambda^2 + 16\lambda + 3 = 0.$$

Straightforward calculations show that for $\lambda > \lambda_0$,

$$\max_{u \in [0, 2\pi]} p_\lambda(u) = p_\lambda(\pi),$$

and this property is preserved for a slight variation of coefficients of $p_\lambda(u)$. Besides, the choice of real initial value $\bar{\Psi}(0)$ implies that $a(t)$ and $\bar{\Psi}(t)$ remain to be real along the optimal trajectory $(a(t), \bar{\Psi}(t))$, and $H(t, a(t), \bar{\Psi}(t), u)$ is a cubic polynomial with respect to $\cos u$. Therefore the real analytic optimal driving function $u(t)$ is equal

to π identically for $t > 0$ small enough. This implies that $u(t) = \pi$ for all $t \geq 0$. It is verified that if $\lambda < \lambda_0$, then

$$\max_{u \in [0, 2\pi]} p_\lambda(u) \neq p_\lambda(\pi).$$

So we have proved the following proposition.

Proposition 1. *Let $\lambda_0 = -0.931\dots$ be the root of equation (24). For $\lambda \geq \lambda_0$, the Koebe function $k(z)$ locally maximizes $\operatorname{Re} \{L\} := \operatorname{Re} \{\lambda a_2 + a_4\}$ on S . For $\lambda < \lambda_0$, $k(z)$ does not maximize $\operatorname{Re} \{L\}$ on S .*

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D. PROKHOROV: DEPARTMENT OF MATHEMATICS AND MECHANICS, SARATOV STATE UNIVERSITY, SARATOV 410012, RUSSIA

E-mail address: ProkhorovDV@info.sgu.ru